Valuation of Exchange Rate Future: When Foreign Exchange Rate and Interest Rates Follow Generalized Jump-Diffusion Process

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Credit Modeling and Incomplete Market
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Summary

This paper derives a closed form pricing formula for foreign exchange (FX) future, where the exchange rate, domestic and foreign interest rate term structure dynamics are governed by a generalized jump-diffusion process, with diffusion process describing continous time price movement, and jump process describing discontinous price movement.

This model captures the real movement of the market. Extreme changes in asset price can be observed due to monetary shocks or catastrophic events in exchange rate and interest rate markets. By assuming the asset price to follow a jump diffusion process, which is geometric Brownian motion plus compound Poisson process for jumps, this model allows discontinuity in the price movement. Also, this paper offers a solution to the problem of the implied volatility smile by the Black-Scholes equation

for derivative pricing. In particular, an implied volatility smile is evidence of the investor's reassessment of the probabilities of fat-tails in asset returns. When jump sizes are set small, a jump-diffusion process can be used to depict fat-tailed distribution.

This paper is the first to price FX future employing a joint jump dynamics of exchange rate, domestic and foreign interest rate process under domestic risk neutral measure. Previous works concentrate on economical meaning of the pricing formula, whereas this paper is computationnaly easier in practice. In fact, the idea behind the pricing formula here is not only applicable to exchange rate, but also to other financial products sensitive to credit risk or other discontinous risk.

The main results of this paper are as follows: (1) Derive jump diffusion process for the exchange rate, domestic and foreign interest rate under domestic risk neutral measure, a method that is inspired by the Duffie-Singleton model (1999); (2) The FX future price is then derived using money market discounting, which is very complex to compute in reality. In order to attain computability of FX future, this paper employs a new measure in which FX future is expressed by a new type of discount function; (3) Relate the pricing formula to practical usage for the insurance agents.

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Contents

1 Introduction

The foreign exchange market is one of the most liquid financial market in the world. Traders include central banks, commercial banks, other financial investors, corporations and individuals. According to the 2016 Bank for International Settlements survey, average foreign exchange daily turnover averaged \$ 5.1 trillion in 2016 (compared to \$1.7 trillion in 1998). \$1.65 trillion was spot transactions and \$3.45 trillion was traded in outright forwards, FX swaps, and other derivatives.

The fact that foreign exchange derivative market is more than twice the spot market indicates that investors tend to hedge in FX derivatives other than holding cash, and that the FX derivative market is so strong that it should have impact on the current market in reverse. This observation is the very initiative for this paper to work on FX derivatives.

Black-Scholes (B-S) equation is the mostly known derivative pricing model. But doubt arises for the "correctness" of setting price evolution process of underlying asset to be continous and normally distributed. These assumptions seem don't hold by market observation.

To elaborate, geometric Brownion motion, a widely used diffusion process, is used to model asset price process in B-S equation. A Brownion motion is a stochastic process W_t with independent, stationary increments dW_t that follow a standard normal distribution N(0,dt). A stochastic process S_t is said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Where μ (the percentage drift) and σ (the percentage volatility) are constant.

An important property of Brownion motion is the continuity of its sample paths, meaning the price fluctuations should remain in a deterministic boundary for every time increment. However, in the case of exchange rate, market observation has shown discontinuity feature, or jump event, in FX movement. Figure 1 (Source: Dukascopy)plots the value of Daily Dollar/Overshore CNY exchange rate from January 2012 to December 2017. In Octorber 2015, the exchange rate undergoes an abrupt upward jump, which shows discontinuities in the price trajectory. This phenomenon can be explained by China central bank's announcement of setting foreign exchange risk reserve ratio from 0 up to 20 % on that day, which greatly increases offshore exchange rate arbitrage costs.

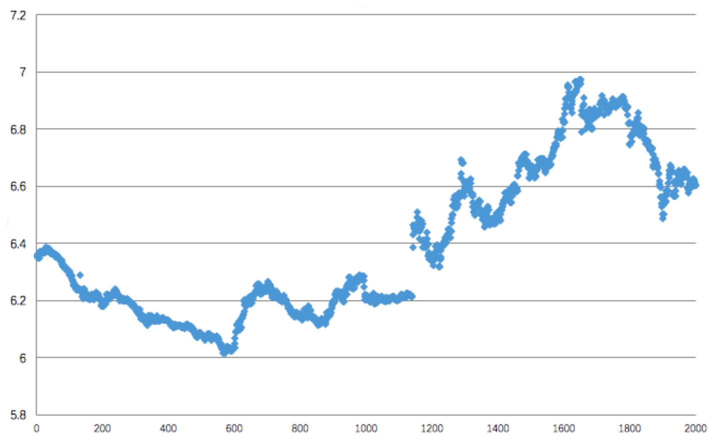


Figure 1: Daily Dollar/Overshore CNY exchange rate from January 2012 to December 2017

Market emotion can also cause price to jump intraday even if the increment time period is set to be small. Figure 2 (Source: Dukascopy) shows 6 minutes tick Dollar/Yen exchange rate at June 23th 2016, when Brexit referendum took places.

Existence of jumps in the sample paths of exchange rates has also been empirically demonstrated in Jorion(1998). Jorion used a mixed ARCH and jump-diffusion model

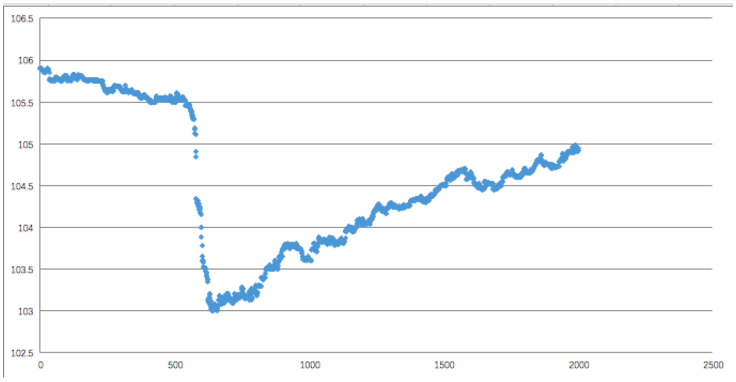
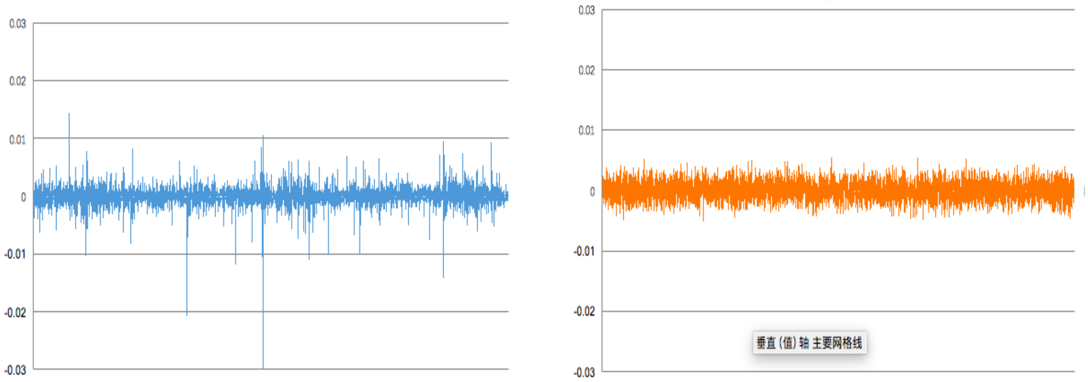


Figure 2: Tick Dollar/Yen exchange rate on June 23th 2016

to show that even after considering conditional heteroskedasticity, there is evidence of significant jumps in the weekly data of US dollar/Deutsche Mark exchange rate.

To capture the cause of jumps in exchange rate, Flood and Hodrick(1986) find jumps in exchange rates are caused mainly by two reasons. One is changes in monetary policies that affect the internal value of a currency, such as the floatation of the Thai baht in 1997 or changes in interest rate. Another is discontinuities in the arrival of news, such as Russian's defaults on its domestic bond debt in 1998; or the 311 earthequake in Japan in 2011, which caused large capital flows among countries.

No only does "jump" represents discontinuity in price movement, asset process with jumps can also model the existence of fat-tailed distribution of returns, as empirically examined in John Cotter(2007). If jump sizes are set to be trivia, return distribution can be expressed as any distribution, if return process are modeled as Brownian motion plus jump process. Figure 3 (Source: Dukascopy) compares the hourly log-returns for Dollar/Yen exchange rate in 2016 to increments of a Brownian motion with the same average volatility. While both return series have the same variance, the Brownian motion model generates returns that have roughly the same sizes, whereas observed Dollar/Yen returns are widely dispersed in their sizes and manifest frequent large peaks.



(a) (b)

Figure 3: (a): Hourly log-returns for Dollar/Yen exchange rate in 2016; (b): log-returns of a Black-Scholes model with same annualized mean and variance

To work on the discontinuity and fat-tailed distribution problem observed in the above cases, mainly two methods are employed in previous works. One method is using a regime-switching model, treating radical effects by monetary policy as a regime change; the other method is to use a jump-diffusion model, by adding jump process to the diffusion process, which is also the method employed in this paper. However, this paper differs from past works by assuming jump-diffusion process not only for exchange rate, but also for interest rate term structure. This paper also allow correlations between exchange rate and interest rate, and introduced risk neutral harzard rate for jumps inspired by Duffie-Singleton's model.

The following part will first discuss about past works treating jumps in exchange rate derivative pricing, and second introduce the proposed model in this paper.

1.1 Past works

Empirically exchange rate movement tend to have discontinuity and fat tails, inconsistent with Black-Scholes model's Brownian motion assumptions for underlying asset movement. The existing academic literature on pricing of foreign exchange derivatives considering jumps can be divided into two categories:

(1) One is to depicit sudden changes in exchange rate with a regime-switching model. Regime-switching model is designed to capture changes in the underlying economic mechanism, which is first presented in Hamilton (1989). In the area of derivative valuation, the regime-switching model fills the gap between fixed volatility in Black-Scholes model and the observed volatility smile, which in the regime-switching framework can be viewed as a special case of a single volatility regime.

For its usage in exchange rate derivative pricing, Dahlquist and Gray (2000) have shown that the changes of regimes is linked with the underlying currency policy alternation, such as switcing from a free float regime to a target zone in the case of European Monetary System. Bollen et al. (2000) value European and American-style options in a regime-switching model, and find out a regime-switching model with independent shifts in mean and variance exhibits a closer fit and more accurate variance forecasts than a range of other models. More recent papers, Goutte et al. (2011) studied foreign exchange rates using a modified Cox-Ingersoll-Ross (CIR) model under a Hamilton Markov regime switching framework, and they illustrate their model on various foreign exchange rate data and clarify some significant economic time periods in which financial or economic crisis appeared, thus, regime switching obtained.

Lin et.al (2013, 2015) proposed a model conbines regime-switching model with jumps, which are to capture risk of sudden news, their empirical results show jumps are attributed to the following three factors: announcement of monetary policies, political risk, and financial crises.

The biggest problem with regime-switching model is that it is a backward method. Using data in the past, parameters in the regime-switching model can be estimated, and different regimes can be recognized. However, in practice, a pricing model is mainly used to determine the asset price in the future. Meanwhile, since FX future are over the counter product, prices are determined by two trading institutions, which have no experience of how to incorporate regime changes into their pricing formula. One reason is that FX future contracts ususally last for a relatively short period from several month to one year, so monetary effect are either forecasted, or hedged. Thus in practice there's little chance of using regime-switching model for derivative pricing.

(2)Another way to deal with jumps is by jump diffusion model, which can be seem as adding a jump process on the traditional diffusion process of asset price movement. The jump process is expressed as a compound Poisson process, which is validated by Levy-Ito's Decomposition. Both diffusion and jump process can be written in different forms.

Bates (1996)'s model accomodates both jump diffusion model and stochastic volatility for pricing American options under systematic jump and volatility risk. The parameters and various submodels are empirically estimated using historical data. He has shown that the stochastic volatility submodel cannot explain the âĂIJvolatility smileâĂİ evidence of implicit excess kurtosis, while jump fears can explain the smile.

Recent papers mostly work on exchange rate options in the field of FX derivatives. Siu et al. (2008) considered pricing currency options under a two-factor Markov modulated jump diffusion volatility model. Bo et al. (2010) discussed a Markov-modulated jump-diffusion modeled by a compound Poisson process for currency option pricing. Rehez et al. (2014) priced foreign exchange option in Jump-Diffusion Model with Cox-Ingersoll-Ross (CIR) interest rate model, by assuming log normal jump sizes and independence between interest rate and exchange rate movement. Option price is derived by no-arbitrage condition in the above works, hence one difficulty is how to create a hedge portfolio when the exchange rate have jumps. If jumps are of random sizes, a riskless hedge portfolio can not be constructed, meaning the traditional hedge technique under no-arbitrage condition cannot be applied to develop the option valuation equation

To deal with the above problem, Merton (1976) assumes that jump risk is nonsystematic, thus the static version of the capital asset pricing model still holds, as risks in jump can be diversified away. However, as the aforementioned Jorion (1988) has proven that exchange rate exhibits systematic discontinuity, capital asset pricing model cannot be applicable when the underlying processes are jump-diffusions, as proved in Jarrow and Rosenfeld (1984).

Another method is to assume type of jump to be finite and parameter λ for Poisson distribution to be deterministic, hence a hedge portfolio can be created and option price is derived under risk neutral measure. The aforementioned papers by Siu et al. (2008), Bo et al. (2010) and Rehez et al. (2014) derive option price in this manner. However, the above assumptions have several limitations. One is that the economical meaning of parameter λ for Poisson distribution under risk neutral measure is unexplained. Under real measure, parameter λ is the event rate and can be measured by counting the average number of jump events in an time interval. But in risk neutral measure, the value of risk neutral λ is unexplained. Another limitation is to assume type of jump to be finite. As studied in AIt-Sahalia and Jacod (2013), empirical tests show the presence of infinite-activity jumps in the high-frequency stock returns. If jumps are infinite, there's no perfect hedge for jump events, thus no hedge portfolio can be constructed. Also, to derive option price under risk neutral probability, the risk free interest rate is used for money market discounting. In the above works, interest rate is set as either constant or stochastic without jumps. For the long-dated products, constant interest rate may hardly reflect the market reality. But if interest rate is set to be stochastic, the pricing formula becomes hardly computable in reality.

This paper works to form the pricing formula that captures the reality and is computable in practice. For the above unrealistic assumptions and computational intractability probelms in deriving foreign exchange option price, this paper only works on foreign exchange future, and leaves the problem of option valuation to furture works.

1.2 Proposed Model

This paper assumes jump happens in three market: exchange rate, domestic and foreign interest rate markets. First we obtain the basic form of jump-diffusion process for these three assets. We use Yen as domestic market, Dollar as foreign market, and exchange from Dollar to Yen.

1.2.1 Jump-diffusion process

To model any distribution process, by Lévy-Itô 's Decomposition, any infinitive divisible random variable S_t can be expressed as:

$$\begin{aligned} \psi_{dS}(\theta) &= \psi_x(\theta)dt \\ \psi_x(\theta) &= \underbrace{\mu\theta + \frac{1}{2}\sigma^2\theta^2}_{\text{continous path}} + \underbrace{\int_{h=-\infty}^{\infty}(e^{h\theta}-1)\lambda(h)dh}_{\text{discontinuous path}} \end{aligned}$$

ψ is cummulant generting function. Stochastic differential equation of S_t becomes

$$\frac{dS_t}{S_t} = \underbrace{\mu dt + \sigma dW_t}_{\text{diffusion process}} + \underbrace{\int_{h=-\infty}^{\infty} h dN(dh)}_{\text{jump process}}$$

By the above equation, a jump-diffusion process of random variable S_t is expressed by two parts: a diffusion process that is geometric Brownian motion with continous sample path; a jump process that is compound Poisson process with discontinous sample path.

To elaborate on the compound Poisson process: a random variable dN_t is said to follow Poisson process as

$$P[dN_t = n|\mathscr{F}_t] = e^{-\lambda_t(dt)}\frac{(\bar{\lambda}_t dt)^n}{n!}$$

Here $\mathscr{F}_t$ means using information up to time t, see Appendix 6.1. Since dt is trivia, by Taylor's expansion and Ito's calculus

$$\begin{cases} P[dN_t = 0|\mathscr{F}_t] = e^{-\bar{\lambda}_t dt} = 1 - \bar{\lambda}_t dt \\ P[dN_t = 1|\mathscr{F}_t] = e^{-\bar{\lambda}_t dt}\bar{\lambda} dt = \bar{\lambda}_t dt \\ P[dN_t \geqslant 2|\mathscr{F}_t] = 0 \end{cases}$$

Thus expectation of Poisson process at time t is

$$E[dN_t|\mathscr{F}_t] = \bar{\lambda}_t dt$$

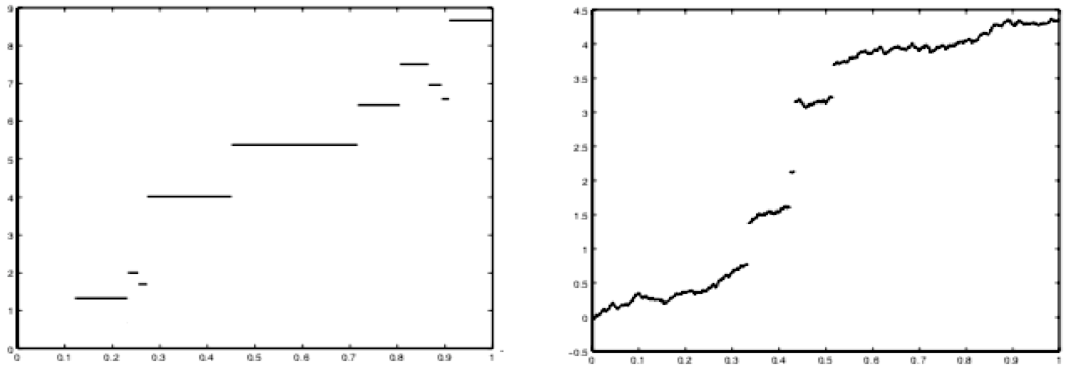
In this study of jump process, for small time interval dt, $\bar{\lambda}_t$ is called the harzard rate, meaning the rate of jump event observed in the market; dN_t is a counting process and is either 0 or 1.

A compound Poisson process X_t is written as integration or sum of Poisson process multiplied by jump size h

$$X_t = \int_{h=-\infty}^{\infty} h dN(dh) = \sum_{i=1}^{\infty} h_i dN(dh_i)$$

Jumps sizes h is independent and identically distributed random variables. For illustrative example, see Figure 4 (Source: US Department of the treasury) for compound Poisson and jump-diffusion process.

The jump-diffusion process of random variable S_t is expressed in the market observed probability. For asset pricing, we need to use risk neutral probability, i.e. change of measure from market observed to risk neutral measure. For diffusion process, change of measure from real to risk neutral probability is done by Black-Scholes as adding a drift term to Brownian motion. Change of measure for jump process is first done by Duffie-Singleton's model (1999) to model default bonds.



(a) (b)

Figure 4: (a) A compound Poisson process with a normal distribution of jump sizes; (b) A jump diffusion process with Brownian motion and compound Poisson process

They derived a risk neutral harzard rate λ, which satisfies martingale condition under risk neutral measure Q as

$$E^Q[dN_t|\mathscr{F}_t] = \lambda_t$$

Duffie also gives economical interpretation of risk neutral harzard rate λ_t: it equals to the credit spread asked between government bond and default bond.

Therefore, jump-diffusion process of random variable S_t in risk neutral measure is expressed as

$$\left.\frac{dS}{S}\right|^Q = rdt + \sigma dW_t + \int_{h=-\infty}^{\infty} h[dN(dh) - \lambda_t(dh)dt]$$

1.2.2 Type of jump

We assume a simultaneous jump event in three markets: the exchange rate, domestic and foreign interest rate market. Jump sizes for each market may vary. If the jump event occurs, for foreign exchange market, the spot exchange rate changes with a jump of a single value, whereas for each interests rate market, the yield curve reshapes, thus jump is a function of term interval.

$$\begin{cases} \text{1.Foreign exchange market: a single jump with jump size } h^x \\ \text{2.Domestic interest rate market: a function of jump that reshapes the entire} \\ \quad \text{yield curve, with jump size } h^{¥}(t,T) \\ \text{3.Foreign interest rate market: a function of jump that reshapes the entire} \\ \quad \text{yield curve, with jump sizes } h^{\$}(t,T) \end{cases}$$

For an illustrative example of jump as a function for the entire yield cure, Figure 5 shows how the result of the presidential election on Nov 8th 2016 triggers jumps on the entire yield curve. Emprical research for yield cure jump has been done by Monika(2001) and Johannes (2004), who argued that kurtosis in short-term interest rates is incompatible with a pure-diffusion model.

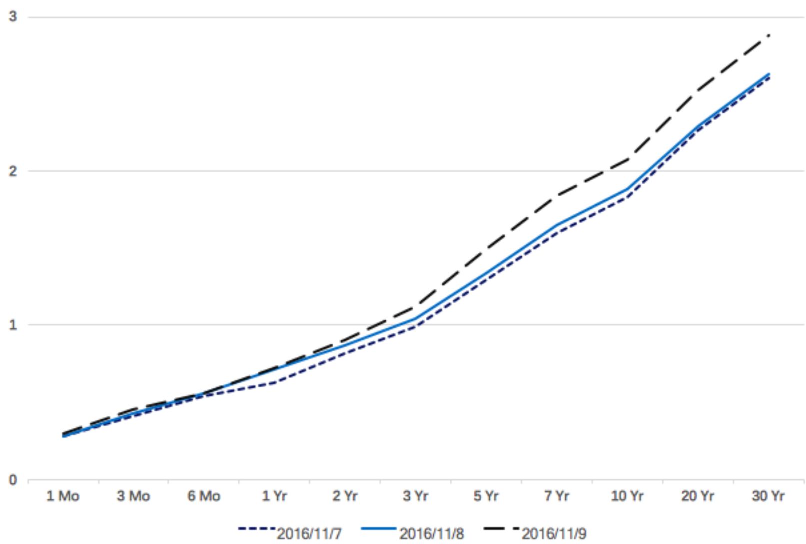


Figure 5: US Treasury Yield curve on Nov 7,8,9 in 2016

In this paper, jump events are categorized by combination of a value (jump size in FX market) and two functions (jump size in interest rate market on the entire yield curve). We call this categorization of combination “a type of jump”. If these combinations or types are finite, then the market can be complete, and jump risks can be eliminated using a hedge portfolio. But in reality market is incomplete. Types of jump are infinite, which is the assumption for this paper. We assume further that these jump events are coming in generalized Poisson process manner. Generalized means arriving intensity, or risk neutral hazard rate λ_t, is future stochastic. This means that the risk neutral harzard rate λ_t is a random variable until immediately before the jump event.

1.2.3 Dynamics of three markets

This paper assumes correlations of price movements between three markets: the exchange rate, domestic interest rate and foreign interest rate market. Results of price dynamics of these three markets under Yen risk neutral measure are shown as follows, which will be further proved in Section 3 and 4:

$$\begin{cases} \text{1.Exchange rate:} \\ \frac{dX^{¥/\$}}{X^{¥/\$}}\Big|^{Q(¥)} = (r^{¥} - r^{\$})dt + \sigma^x dW^x + \int_\omega h^x \Big\{ dN(d\omega) - \lambda_t(d\omega)dt \Big\} \\ \text{2.Yen forward rate:} \\ df^{¥}(t,T)\Big|^{Q(¥)} = \vec{\sigma}^{¥}(t,T) \int_t^T \vec{\sigma}^{¥}(t,\dot{T})d\dot{T}dt + \vec{\sigma}^{¥}(t,T)d\vec{W}_t^{¥} \\ \qquad\qquad + \int_\omega h^{¥}(t,T) \Big\{ dN_t(d\omega) - e^{-\int_t^T h^{¥}(t,\dot{T})d\dot{T}} \lambda_t(d\omega)dt \Big\} \\ \text{3.Dollar forward rate:} \\ df^{\$}(t,T)\Big|^{Q(¥)} = \vec{\sigma}^{\$}(t,T) \Big(\int_t^T \vec{\sigma}^{\$}(\tau,\dot{T})d\dot{T} - \sigma^x \vec{\rho}_t \Big) dt + \vec{\sigma}^{\$}(t,T)d\vec{W}_t^{\$} \\ \qquad\qquad + \int_\omega h^{\$}(t,T) \Big\{ dN_t(d\omega) - e^{-\int_t^T h^{\$}(t,\dot{T})d\dot{T} + log(1+h^x)} \lambda_t(d\omega)dt \Big\} \end{cases}$$

Where dW^x is Brownian motion for exchange rate process, $d\vec{W}_t^{¥}$ and $d\vec{W}_t^{\$}$ are multi-dimensional Brownian motion for Yen and Dollar forward rate process.

Correlations between the exchange rate and the interest rates are expressed as correlations between Brownian motions

$$\begin{aligned} dW_t^x d\vec{W}_t^{¥} &= \vec{\rho}_t dt \\ d\vec{W}_t^{¥} \otimes d\vec{W}_t^{\$} &= P_t^{¥\otimes\$} dt \end{aligned}$$

where $\vec{\rho}_t$ is correlation vector between exchange rate and Yen forward rate, and $P_t^{¥\otimes\$}$ is correlation martix between Yen and Dollar forward rate.

There are several merits of writing price moves in the above jump-diffusion model. First, it captures market reality. It allows sample path discontinuity and different distributions of price movement. It assumes the market is incomplete and allows correlations between different markets as observed in reality. See Table 1 for a comparison between pure diffusion and jump-diffusion models.

Table 1: Modelling price moves: Pure diffusion models vs. Jump-diffusion models

	Diffusion models	Jump-diffusion models
Continuity	Price moves are continous	Jumps/ discontinuities in price moves
	Large sudden moves do not occur	Arise to large losses
Volatility	Fixed or stochastic	Generic property
Market Completeness	Market is complete	Market is incomplete
Derivative Pricing	Create hedge portfolios	Hedge portfolios don't exist

Second, it obtains risk neutral expression for jump process of each market, and gives economical meaning of risk neutral harzard rate $\bar{\lambda}$. In Duffie-Singleton's model, risk neutral harzard rate λ_t is expressed as credit spread between default and government bond. This credit spread can be get from the market and covers risk premium for default(jump) events plus operational cost of institutions who sell this default bond. In the case of jumps in exchange rate and interest rate market, we assume the existence of insurance agents, who are willing to sell insurance products that covers the jump events. Insurance agents are the best to handle jumps, as their business nature is to price jumps. Mortality, adverse weather conditions, or corporate default, can all be seem as jump events. The market observed harzard rate $\bar{\lambda}_t$ corresponds to the pure insurance fee that is the actual probability of the jump event. The risk neutral harzard rate λ_t under risk neutral measure is the regular insurance fee, which is the pure insurance fee plus operational cost of the insurance agent. Thus risk neutral harzard rate λ_t equals to the insurance fee we are charged by the insurance agent in the market. In fact, there were insurance agencies like American International Group (AIG) who sold credit risk insurance products, and their insurance fee is the risk neutral harzard rate λ_t for default(jump) event on subprime mortgage.

1.2.4 A new discount function for FX future

The pricing formula for foreign exchange future can be expressed as a discount function of future price movements. However, as interest rate is modeled by the jump-diffusion model in this work, the discount function(discount by instantaneous interest rate) becomes extremely difficult to compute, as the interest rate changes at each time point. By change of measure, this paper obatins a new discount function that is much easier to compute, thus derives a closed-form formula for FX future prices.

In summary, the pricing formula for FX future obtained in this paper differs from past works by assuming: 1. The market is incomplete, i.e, we can't create hedge portfolio to price derivatives; 2. Exchange rate, domestic and foreign interest rate movements are correlated, which is often observed in the market. 3. Jump process in FX future is a combination of different jump sizes in three markets: jumps in FX market change the value of spot exchange rate, and jumps in domestic and foreign interest rate markets change the entire shape of yield curve. 4. Jumps in three market are set to be sychronized by the same triggering events, but jump sizes can be different.

2 The Foreign Exchange Rate Model

2.1 Interest Rate Parity

This paper uses the Interest rate parity (IRP) to explain the value and movements of exchange rates. According to IRP, there will be no arbitrage opportunity in interest rate differentials between two currencies. IRP assumes that the actions of international investors, motivated by cross-country differences in interest rates, i.e. rates of return on government bonds, induce changes on the spot exchange rate. For example, set domestic currency as Japanese Yen, and foreign currency as US Dollar. To get 1$ payoff at time T, by IRP, the investor invests the same amount of Yen either in the Dollar or Yen market at time t. See Figure 6.

Figure 6: Interest rate parity

$$\begin{cases} \text{1. If invested in Dollar Market: 1\$ at time T equals to } D^{\$}(t,T) \text{ amount of pure} \\ \quad \text{discount Dollar bond at time t, exchange into Yen at exchange rate } X_t^{¥/\$} \\ \text{2. If invested in Yen Market: exchange 1\$ at T into Yen at exchange rate } X_T^{¥/\$}, \\ \quad \text{and discounted by Yen risk free rate } r^{¥} \text{ to time t} \end{cases}$$

Investment 1 and 2 should have the same present value at time t, otherwise there will be an arbitrage opportunity and spot exchange rate will move until the arbitrage opportunity disappears. Express the two investments in domestic (Yen) risk neutral measure,

$$X_t^{¥/\$} D^{\$}(t,T) = E^{Q(¥)}[e^{-\int_t^T r^{¥} dt} X_T^{¥/\$} | \mathscr{F}_t] \tag{1}$$

Where $X_t^{¥/\$}$ is the spot exchange rate between Yen and Dollar at time t; $D^{\$}(t,T)$ is the value of pure discount Dollar bond at time t with term T; $r^{¥}$ is the instantaneous Yen risk free rate; $Q(¥)$ means Yen risk neutral measure, and $\mathscr{F}_t$ means using information up to time t.

Set $Q(\$)$ as Dollar risk neutral measure. In Dollar risk neutral measure, by defination

$$D^{\$}(t,T) = E^{Q(\$)}[e^{-\int_t^T r^{\$} dt} | \mathscr{F}_t]$$

Where $r^{\$}$ is the instantaneous Dollar risk free rate. Set T as a small change from t, i.e. T=t+dt. e.q.(1) becomes

$$X_t^{¥/\$} E^{Q(\$)}[e^{r^{\$} dt} | \mathscr{F}_t] = E^{Q(¥)}[e^{r^{¥} dt}(X_t^{¥/\$} + dX) | \mathscr{F}_t]$$

Since $r^{\$}$ is no longer random variable at time t, $E^{Q(\$)}[e^{r^{\$}dt}|\mathscr{F}_t] = e^{r^{\$dt}}$

$$X_t^{¥/\$} = E^{Q(¥)}\left[e^{-(r^{¥}-r^{\$})dt}(X_t^{¥/\$} + dX^{¥/\$})\Big|\mathscr{F}_t\right]$$

By Taylor's expansion and Ito's calculus

$$\begin{aligned} X_t^{¥/\$} &= E^{Q(¥)}\left[\left(1-(r^{¥}-r^{\$})dt\right)\left(X_t^{¥/\$} + dX^{¥/\$}\right)|\mathscr{F}_t\right] \\ &= X_t^{¥/\$} - (r^{¥}-r^{\$})X_t^{¥/\$}dt + E^{Q(¥)}\left[dX^{¥/\$}\big|\mathscr{F}_t\right] \end{aligned}$$

Thus

$$E^{Q(¥)}\left[\frac{dX^{¥/\$}}{X^{¥/\$}}\Big|\mathscr{F}_t\right] = (r^{¥}-r^{\$})dt \tag{2}$$

2.2 Jump-diffusion process for exchange rate

The diffusion process for exchange rate by eq.(2) is set as

$$\frac{dX^{¥/\$}}{X^{¥/\$}} = (r^{¥}-r^{\$})dt + \sigma_t^x dW^x$$

Where σ_t^x is the variance of exchange rate, and can be set as stochastic or constant. dW^x is a Brownion motion under Yen risk neutral measure.

For jump process, by Lévy Itô's Decomposition, observed jump process can be written as a componded Poisson distribution. But since we use Yen risk neutral measure instead of real probability here, the jump process is set to be in the following form,

$$\int_{\omega} h_t^x \left\{dN(d\omega) - \lambda_t(d\omega)dt\right\}$$

$$
\begin{cases}
h_t^x\text{: the jump size at time t;} \\
\mathrm{d}N\text{: counting of instantaneous jump for small time period dt. } \mathrm{d}N \text{ is 0 or 1 i.e.} \\
\quad\text{no jump or only one jump, as higher numbers of jumps can be ignored as} \\
\quad\text{probability being negligible;} \\
\lambda_t\text{: time-variable harzard rate at time t under Yen risk neutral measure, which} \\
\quad\text{is defined as } E^{Q(¥)}[dN|\mathscr{F}_t] = \lambda_t dt; \\
\omega\text{: a sample corresponded to the jump size } h_t^x\text{. Probability can be defined by} \\
\quad\text{associating a sample to a type of jump.}
\end{cases}
$$

Since here we need to consider three types of jumps in three different markets in dynamics, ω is used to specify a sample combination of jumps from three markets.This is a marginal sample in fact; There are two kind of randomness associated with a jump. One is whether the jump occurs or not. The other is which type of jump takes place. This particular sample ω is from the latter one.

Combining diffusion process and jump process, the jump diffusion process of exchange rate under Yen risk neutral probability is written as

$$\frac{dX^{¥/\$}}{X^{¥/\$}} = (r^{¥} - r^{\$})dt + \sigma^x dW^x + \int_\omega h_t^x \Big\{ dN(d\omega) - \lambda_t(d\omega)dt \Big\} \tag{3}$$

The jump process is in the above form because by eq.(2)

$$E^{Q(¥)}\Big[\frac{dX^{¥/\$}}{X^{¥/\$}}\Big] = (r^{¥} - r^{\$})dt \tag{4}$$

Take expectation of eq.(3) under Yen risk neutral measure

$$E^{Q(¥)}[\frac{dX^{¥/\$}}{X^{¥/\$}}|\mathscr{F}_t] = (r^{¥} - r^{\$})dt + E^{Q(¥)}[\text{ Brownian motion}|\mathscr{F}_t] + E^{Q(¥)}[\text{jump}|\mathscr{F}_t] \tag{5}$$

eq.(4) and eq.(5) is equal.

For eq.(5), since $E^{Q(¥)}[\text{Brownian motion}|\mathscr{F}_t] = E^{Q(¥)}[\sigma_t^x dW^x|\mathscr{F}_t]$=0,

$$E^{Q(¥)}[\text{jump}|\mathscr{F}_t] = 0$$

For Poisson process, $E^{Q(¥)}[dN(d\omega)|\mathscr{F}_t] = \lambda_t(d\omega)dt$. Thus jump process under Yen risk neutral measure must be in form of:

$$\int_\omega h_t^x \left\{dN(d\omega) - \lambda_t(d\omega)dt\right\}$$

To be more precise, the above process should be called jump process martingale, as it is composed of a compound Poisson process(Jump process) minus a drift term, which makes the process expectation to be zero under Yen risk neutral measure. But for the simplicity of wording here, we still call the above process as jump process.

3 Foreign exchange future pricing formula

A foreign exchange future is a contract to exchange one currency for another at a specified date in the future at a price (exchange rate) that is fixed on the purchase date.

Foreign exchange future H(t,T) written in Yen risk neutral measure is

$$H(t,T) = E^{Q(¥)}\left[X_T^{¥/\$}\middle|\mathscr{F}_t\right]$$

Since $E^{Q(¥)}\left[\frac{X_T^{¥/\$}}{H(t,T)}\middle|\mathscr{F}_t\right] = 1$, by Girsanov's Theorem, a new measure $H_T(¥)$ can be defined as

$$E^{H_T(¥)}\left[S\middle|\mathscr{F}_t\right] = E^{Q(¥)}\left[S\frac{X_T^{¥/\$}}{H(t,T)}\middle|\mathscr{F}_t\right] \tag{6}$$

We call $H_T(¥)$ as Yen future measure. Recall eq.(1)

$$E^{Q(¥)}\left[e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}\middle|\mathscr{F}_t\right] = X_t^{¥/\$}D^{\$}(t,T)$$

Compared eq.(1) with eq.(6)

$$E^{H_T(¥)}\left[e^{-\int_t^T r^{¥}dt}\middle|\mathscr{F}_t\right] = E^{Q(¥)}\left[\frac{e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}}{H(t,T)}\middle|\mathscr{F}_t\right] = \frac{X_t^{¥/\$}D^{\$}(t,T)}{H(t,T)} = D^{H_T(¥)}(t,T) \tag{7}$$

Call $D^{H_T(¥)}(t,T)$ a discount Yen bond future. Compare $D^{H_T(¥)}(t,T)$ with the pure Dollar discount bond $D^{\$}(t,T)$, which can be expressed by forward Dollar rate $f^{\$}(t,T)$ as

$$D^{\$}(t,T) = e^{-\int_t^T f^{\$}(t,\dot{T})d\dot{T}}$$

Set a Yen forward rate future $f^{H_T(¥)}(t,T)$ be so that

$$D^{H_T(¥)}(t,T) = e^{-\int_t^T f^{H_T(¥)}(t,\dot{T})d\dot{T}} \tag{8}$$

Replace eq.(7) with eq.(8)

$$H(t,T) = \frac{X_t^{¥/\$} D^{\$}(t,T)}{D^{H_T(¥)}(t,T)} = \frac{X_t^{¥/\$} D^{\$}(t,T)}{e^{-\int_t^T f^{H_T(¥)}(t,\dot{T})d\dot{T}}}$$

Since $X_t^{¥/\$}$ and $D^{\$}(t,T)$ are known from the market, in order to find FX future price H(t,T), we only need to solve for $f^{H_T(¥)}(t,T)$ by the above equation. To calculate $f^{H_T(¥)}(t,T)$, differentiating eq.(7) and eq.(8) in terms of T

$$\begin{aligned}
e^{-\int_t^T f^{H_T(¥)}(t,\dot{T})d\dot{T}} f^{H_T(¥)}(t,T) &= E^{H_T(¥)}\left[r^{¥} e^{-\int_t^T r^{¥} dt} \middle| \mathscr{F}_t \right] \\
f^{H_T(¥)}(t,T) &= E^{H_T(¥)}\left[r^{¥} \frac{e^{-\int_t^T r^{¥} dt}}{D^{H_T(¥)}(t,T)} \middle| \mathscr{F}_t \right] \\
&= E^{Q(¥)}\left[r^{¥} \frac{e^{-\int_t^T r^{¥} dt}}{D^{H_T(¥)}(t,T)} \frac{X_T^{¥/\$}}{H(t,T)} \middle| \mathscr{F}_t \right] \\
&= E^{Q(¥)}\left[r^{¥} \frac{e^{-\int_t^T r^{¥} dt} H(t,T)}{X_t^{¥/\$} D^{\$}(t,T)} \frac{X_T^{¥/\$}}{H(t,T)} \middle| \mathscr{F}_t \right] \\
&= E^{Q(¥)}\left[r^{¥} \frac{e^{-\int_t^T r^{¥} dt} X_T^{¥/\$}}{X_t^{¥/\$} D^{\$}(t,T)} \middle| \mathscr{F}_t \right]
\end{aligned}$$

Therefore

$$f^{H_T(¥)}(t,T) = E^{Q(¥)}\left[r^{¥} \frac{e^{-\int_t^T r^{¥} dt} X_T^{¥/\$}}{X_t^{¥/\$} D^{\$}(t,T)} \middle| \mathscr{F}_t \right] \tag{9}$$

Theoretically, the above expectation equation for $f^{H_T(¥)}(t,T)$ is computable, thus we can solve for H(t,T). But this paper has set the interest rate $r^{¥}$ to be jump diffusion process. For different time t, $r^{¥}$ is different random variable. To evaluate expectation form, we need to find infinite dimensional probability density function, which is not possible to compute. And as mentioned in the introduction part, past works assuming stochastic interest rate have provided pricing formula in above expectation form, which is mathematically succinct and economically comprehensive, but computationally hard.

This paper tries to find a different way to express for $f^{H_T(¥)}(t,T)$, which is desired to be computable with a closed form solution. The intuition comes from HJM's model, which models forward interest rate in forward measure, and gives both economical and computational expression. Recall in HJM, the above expectation form, i.e, $E^{Q(¥)}[e^{-\int_t^T r^{¥}dt}|\mathscr{F}_t]$ is computable as

$$\begin{cases} E^{Q(¥)}[e^{-\int_t^T r^{¥}dt}|\mathscr{F}_t] = D^{¥}(t,T) = e^{-\int_t^T f^{¥}(t,\dot{T})d\dot{T}} \\ \qquad f^{¥}(t,T) = E^{Q(¥)}\left[r_T^{¥}\dfrac{e^{-\int_t^T r^{¥}dt}}{D^{¥}(t,T)}\middle|\mathscr{F}_t\right] = E^{F_T(¥)}\left[r_T^{¥}\middle|\mathscr{F}_t\right] \end{cases} \tag{10}$$

Where $F_T(¥)$ represents Dollar forward measure and is computable from relationship between forward and spot interest rate. By changing from spot rate to forward rate, the money market discount $e^{-\int_t^T r^{¥}dt}$ is much easier to compute, as forward rate $f^{¥}(t,T)$ is a single expectation of a single variable.

Drawing an analogy between eq. (9) with eq.(10), $f^{H_T(¥)}(t,T)$ can be computed if we can change to a new measure S as

$$\begin{cases} E^{H_T(¥)}[e^{-\int_t^T r^{¥}dt}|\mathscr{F}_t] = D^{H_T(¥)}(t,T) = e^{-\int_t^T f^{H_T(¥)}(t,\dot{T})d\dot{T}} \\ \qquad f^{H_T(¥)}(t,T) = E^{Q(¥)}\left[r^{¥}\dfrac{e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}}{X_t^{¥/\$}D^{\$}(t,T)}\middle|\mathscr{F}_t\right] = E^{S}\left[r_T^{¥}\middle|\mathscr{F}_t\right] \end{cases} \tag{11}$$

S is an unknown new measure, and eq. (11) can only be true by Girsanov's Theorom

$$if \quad E^{Q(¥)}\left[\frac{e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}}{X_t^{¥/\$}D^{\$}(t,T)}\middle|\mathscr{F}_t\right] = 1$$

Recall e.q.(1) as

$$X_t^{¥/\$} D^{\$}(t,T) = E^{Q(¥)}[e^{-\int_t^T r^{¥} dt} X_T^{¥/\$} | \mathscr{F}_t]$$

Since $X_t^{¥/\$}$ and $D^{\$}(t,T)$ are known at time t, proved

$$E^{Q(¥)}\left[\frac{e^{-\int_t^T r^{¥} dt} X_T^{¥/\$}}{X_t^{¥/\$} D^{\$}(t,T)} \middle| \mathscr{F}_t\right] = 1 \tag{12}$$

Thus e.q.(11) is proved to be true. Now what we need to find is what is the new measure S.

3.1 The new measure S

By eq.(12)

$$E^{Q(¥)}\left[\frac{e^{-\int_t^T r^{¥} dt} X_T^{¥/\$}}{X_t^{¥/\$}} \middle| \mathscr{F}_t\right] = D^{\$}(t,T) = E^{Q(\$)}\left[e^{-\int_t^T r^{\$} dt} \middle| \mathscr{F}_t\right] \tag{13}$$

$D^{\$}(t,T)$ can also be expressed by Dollar forward rate $f^{\$}(t,T)$ as

$$D^{\$}(t,T) = e^{-\int_t^T f^{\$}(t,\dot{T}) d\dot{T}} \tag{14}$$

Differentiate eq.(13) and eq.(14) in terms of T:

$$\begin{aligned} -e^{-\int_t^T f^{\$}(t,\dot{T}) d\dot{T}} f^{\$}(t,T) &= E^{Q(\$)}\left[-r^{\$} e^{-\int_t^T r^{\$} dt} \middle| \mathscr{F}_t\right] \\ f^{\$}(t,T) &= E^{Q(\$)}\left[\frac{e^{-\int_t^T r^{\$} dt}}{D^{\$}(t,T)} r^{\$} \middle| \mathscr{F}_t\right] = E^{F_T(\$)}\left[r^{\$} \middle| \mathscr{F}_t\right] \end{aligned} \tag{15}$$

Where $F_T(\$)$ represents Dollar forward measure by the defination of forward rate

$f^{\$}(t,T)$. Combining eq.(13) and eq.(15),

$$E^{Q(¥)}\left[\frac{e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}}{D^{\$}(t,T)X_t^{¥/\$}}r^{\$}\middle|\mathscr{F}_t\right]=E^{Q(\$)}\left[\frac{e^{-\int_t^T r^{\$}dt}}{D^{\$}(t,T)}r^{\$}\middle|\mathscr{F}_t\right]=E^{F_T(\$)}\left[r^{\$}\middle|\mathscr{F}_t\right] \tag{16}$$

Therefore, it has beem proved that the new measure S is the Dollar forward measure $F_T(\$)$.

3.2 Standard form of Radon-Nikodym Derivative P

Although we have proved the new measure to be Dollar forward measure $F_T(\$)$, in order to find relationship between jump-diffusion process in each measure, we need to know the standard form of Radon-Nikodym Derivative P, as proved in Appendix. Recall eq.(11) that

$$f^{H_T(¥)}(t,T)=E^{Q(¥)}\left[r^{¥}\frac{e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}}{X_t^{¥/\$}D^{\$}(t,T)}\middle|\mathscr{F}_t\right]=E^{Q(¥)}\left[r^{¥}P\middle|\mathscr{F}_t\right]=E^{F_T(\$)}\left[r^{¥}\middle|\mathscr{F}_t\right]$$

The Radon-Nikodym Derivative P between between Yen risk neutral measure $Q(¥)$ and the Dollar forward measure $F_T(\$)$ is

$$P=\frac{e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}}{X_t^{¥/\$}D^{\$}(t,T)}$$

The above Radon-Nikodym Derivative P is expressed in economical form, but we need to find the standard form of the Radon-Nikodym Derivative P between Dollar forward measure $F_T(\$)$ and Yen risk neutral measure $Q(¥)$.

Change of measure from Dollar forward measure $F_T(\$)$ and Yen risk neutral measure $Q(¥)$ can be devided into two steps: 1. change of measure from Dollar forward measure $F_T(\$)$ to Dollar risk neutral measure $Q(\$)$; 2. change of measure from Dollar risk neutral measure $Q(\$)$ to Yen risk neutral measure $Q(¥)$

As demonstrated in HJM model, standard form of Radon-Nikodym Derivative

This is the very reason why we use forward rate in this paper.

Here we make an analogy to HJM model again. Recall in HJM model, forward rate is expressed as

$$f^{\$}(t,T) = E^{Q(\$)}\left[r^{\$} \frac{e^{-\int_t^T r^{\$} dt}}{D^{\$}(t,T)} \middle| \mathscr{F}_t \right] = E^{F_T(\$)}\left[r^{\$} \middle| \mathscr{F}_t \right]$$

The standard form of Radon-Nikodym Derivative can be obtained as

$$\frac{e^{-\int_t^T r^{\$} dt}}{D^{\$}(t,T)} =$$

Where the standard form of Radon-Nikodym Derivative $\tilde{P}$ between Yen forward measure and Yen risk neutral measure is derived. This give us the intuition to derive standard form of P by first changing from Dollar forward measure to Dollar risk neutral measure by HJM, and then changing from Dollar risk neutral measure to Yen risk neutral measure by eq.(**??**).

If not considering jumps, Dollar forward rate process under Dollar forward measure under HJM framework is expressed as

$$df^{\$}(t,T)\Big|^{F_T(\$)} = \vec{\sigma}^{\$}(t,T) d\vec{W}_t^{\$}$$

$\vec{\sigma}^{\$}(t,T)$ is the vector volatility for Dollar forward rate, $d\vec{W}_t^{\$}$ is a multidimentional Brownion motion vector following N($\vec{0}$,Idt), I is a identity matrix. To add jump process into the equation, recall that jump martingale process written in Yen risk neutral measure is

$$\int_\omega h^\$(t,T)\{dN_t(d\omega)-\lambda_t(d\omega)dt\}$$

Where $h^\$(t,T)$ represents jump size on the yield curve. Since HJM is written in Dollar forward measure, to make jump martingale in forward measure, we need to multiply a number, say $e^{\nu(t,T)}$, to risk neutral harzard rate λ_t, to make expectation of jump martingale process under Dollar forward measure to be zero.

Therefore, Dollar forward rate in Dollar forward measure under HJM framework with jumps is

$$df^\$(t,T)\Big|^{F_T(\$)} = \vec{\sigma}^\$(t,T)d\vec{W}_t^\$ + \int_\omega h^\$(t,T)\Big\{dN_t(d\omega)-e^{\nu(t,T)}\lambda_t(d\omega)dt\Big\} \tag{17}$$

3.2.1 From Dollar forward measure to Dollar risk neutral measure

Integrate eq.(17) in terms of t from t to T

$$\begin{aligned} & f^\$(T,T)\Big|^{F_T(\$)} = r_T^\$ \\ = \quad & f^\$(t,T) + \int_{\tau=t}^T \vec{\sigma}^\$(t,T)d\vec{W}_\tau^\$ + \int_t^T\int_\omega h^\$(t,T)dN_t(d\omega) - \int_t^T\int_\omega e^{\nu(\tau,T)}\lambda_\tau(d\omega)d\tau \end{aligned}$$

Futher intergrate in terms of T from (t,T)

$$\begin{aligned} \int_t^T r_{\dot{T}}^\$ d\dot{T} \quad = \quad & \int_t^T f^\$(t,\dot{T})d\dot{T} + \int_t^T\int_{\tau=t}^T \vec{\sigma}^\$(t,\dot{T})d\dot{T}d\vec{W}_\tau^\$ \\ + \quad & \int_t^T\int_\omega\int_t^T h^\$(t,\dot{T})d\dot{T}dN_t(d\omega) - \int_t^T\int_\omega\int_t^T e^{\nu(\tau,\dot{T})}\lambda_\tau(d\omega)d\dot{T}d\tau \end{aligned}$$

Take exponential at both side:

$$
\begin{aligned}
&\frac{e^{-\int_t^T r^{\$}_{\dot{T}} d\dot{T}}}{e^{-\int_t^T f^{\$}(t,\dot{T}) d\dot{T}}} = \frac{e^{-\int_t^T r^{\$}_{\dot{T}} d\dot{T}}}{D^{\$}(t,T)} \\
&= \exp\Big\{ -\int_t^T \int_{\tau=t}^T \vec{\sigma}^{\$}(t,\dot{T}) d\dot{T} d\vec{W}^{\$}_{\tau} + \int_t^T \int_{\omega} \int_t^T e^{\nu(\tau,\dot{T})} \lambda_{\tau}(d\omega) d\dot{T} d\tau \\
&- \int_t^T \int_{\omega} \int_t^T h^{\$}(t,\dot{T}) d\dot{T} dN_t(d\omega) \Big\}
\end{aligned} \tag{18}
$$

By Girsanov Theorm, for standard form of Radon-Nikodym derivative for Brownion motion is expressed as

$$
E^{Q(\$)}[e^{\vec{a}d\vec{W} - \frac{1}{2}||\vec{a}||^2 dt} d\vec{W}] = E^{Q(\$)}[d\vec{W} + \vec{a}dt] = E^{F_T(\$)}[d\tilde{\vec{W}}]
$$

Recall in eq.(16)

$$
E^{Q(\$)}\left[\frac{e^{-\int_t^T r^{\$} dt}}{D^{\$}(t,T)} r^{\$} \middle| \mathscr{F}_t \right] = E^{F_T(\$)}\left[r^{\$} \middle| \mathscr{F}_t \right]
$$

Thus

$$
\frac{e^{-\int_t^T r^{\$} dt}}{D^{\$}(t,T)} = e^{\vec{a}d\vec{W} - \frac{1}{2}||\vec{a}||^2 dt} \tag{19}
$$

And the ralationship between $d\tilde{\vec{W}}$ and $d\vec{W}$ is

$$
\begin{aligned}
d\vec{W}^{\$}_t &\sim N(0, dt) \\
d\tilde{\vec{W}}^{\$}_t &\leftarrow d\vec{W}^{\$}_t - \vec{a}(t,T)dt
\end{aligned}
$$

The problem left is how to solve for $\vec{a}(t,T)$. In Dollar risk neutral measure, Dollar forward rate process becomes

$$df^{\$}(t,T)\Big|^{Q(\$)} = \vec{\sigma}^{\$}(t,T)d\tilde{\vec{W}}_t^{\$} = \vec{\sigma}^{\$}(t,T)(d\vec{W}_t^{\$} - \vec{a}(t,T)dt) \tag{20}$$

Intergrating eq.(20) in terms of t from t to T

$$f^{\$}(T,T)\Big|^{Q(\$)} = r_T^{\$}\Big|^{Q(\$)} = f^{\$}(t,T) + \int_t^T \vec{\sigma}^{\$}(\tau,T)\vec{a}(t,T)d\tau + \int_{\tau=t}^T \vec{\sigma}^{\$}(t,T)d\vec{W}_\tau^{\$} \tag{21}$$

Futher intergrating eq.(21) in terms of from t to T

$$\int_t^T r_{\dot{T}}^{\$} d\dot{T}\Big|^{Q(\$)} = \int_t^T f^{\$}(t,\dot{T})d\dot{T} + \int_t^T \int_\tau^T \vec{\sigma}^{\$}(\tau,\dot{T})\vec{a}(\tau,\dot{T})d\dot{T}d\tau + \int_{\tau=t}^T \int_\tau^T \vec{\sigma}^{\$}(t,\dot{T})d\dot{T}d\vec{W}_\tau^{\$}$$

Take exponential both side and compared with eq.(19)

$$\begin{aligned}
\frac{e^{-\int_t^T r^{\$}dt}}{D^{\$}(t,T)}\Big|^{Q(\$)} &= \exp\Big[-\int_t^T \int_\tau^T \vec{\sigma}^{\$}(\tau,\dot{T})\vec{a}(\tau,\dot{T})d\dot{T}d\tau - \int_{\tau=t}^T \int_\tau^T \vec{\sigma}^{\$}(t,\dot{T})d\dot{T}d\vec{W}_\tau^{\$}\Big] \\
&= \exp\Big\{\vec{a}(t,T)d\vec{W}^{\$} - \frac{1}{2}||\vec{a}(t,T)||^2 dt\Big\}
\end{aligned}$$

we have

$$-\int_t^T \int_{\tau=t}^T \vec{\sigma}^{\$}(t,\dot{T})d\dot{T}d\vec{W}_\tau^{\$} = \vec{a}(t,T)d\vec{W}^{\$}$$

Thus

$$\vec{a}(t,T) = -\int_t^T \vec{\sigma}^{\$}(\tau,\dot{T})d\dot{T}$$

For jump process, the Standard form of Radon-Nikodym derivative is

$$E[dN\cdot e^{logQdN-(Q-1)dt}|\mathscr{F}_t] = \tilde{E}[d\tilde{N}|\mathscr{F}_t]$$

Compare with eq.(18), we have

$$e^{logQdN} = exp\left\{-\int_t^T \int_t^T h^{\$}(t,\dot{T})d\dot{T}dN\right\}$$

Thus

$$Q = -\int_t^T h^{\$}(t,\dot{T})d\dot{T}$$

Therefore, Dollar interest rate process in Dollar risk neutral measure is

$$\begin{aligned} df^{\$}(t,T)\Big|^{Q(\$)} &= \vec{\sigma}^{\$}(t,T)\int_t^T \vec{\sigma}^{\$}(t,\dot{T})d\dot{T}dt + \vec{\sigma}^{\$}(t,T)d\vec{W}_t^{\$} \\ &+ \int_{\omega} h^{\$}(t,T)\left\{dN_t(d\omega) - e^{-\int_t^T h^{\$}(t,\dot{T})d\dot{T}}\lambda_t(d\omega)dt\right\} \end{aligned}$$

3.2.2 From Dollar risk neutral measure to Yen risk neutral measure

By Girsonov's Theorem for Brownion motion

$$E[e^{adW-\frac{1}{2}a^2dt}d\vec{W}] = E[d\vec{W} + a\vec{\rho}dt] = \tilde{E}[d\tilde{\vec{W}}]$$

We already have

$$E^{Q(¥)}\left[\frac{e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}}{X_t}\middle|\mathscr{F}_t\right] = D^{\$}(t,T) = E^{Q(\$)}\left[e^{-\int_t^T r^{\$}dt}\middle|\mathscr{F}_t\right]$$

Thus for Radon-Nikodym derivative

$$e^{adW-\frac{1}{2}a^2dt} = \frac{e^{-\int_t^T r^{¥}dt}X_T^{¥/\$}}{X_t^{¥/\$}e^{-\int_t^T r^{\$}dt}}$$

And relationship between $d\tilde{\vec{W}}$ and $d\vec{W}$

$$\begin{aligned} d\vec{W}^{\$} &\sim N(0, dt) \\ d\tilde{\vec{W}}^{\$} &\leftarrow d\vec{W}^{\$} - a\vec{\rho}dt \end{aligned}$$

To calculate Radon-Nikodym derivative, taking logarithm of exchange rate

$$\begin{aligned} dlogX^{¥/\$} &= log(X^{¥/\$} + dX^{¥/\$}) - logX^{¥/\$} = dlog(1 + \frac{dX^{¥/\$}}{X^{¥/\$}}) \\ &= \left\{\frac{dX^{¥/\$}}{X^{¥/\$}} - \frac{1}{2}(\frac{dX^{¥/\$}}{X^{¥/\$}})^2\right\}\Big|_{no\ jump} + \int_\omega \Big[log(X^{¥/\$} + X^{¥/\$}h^x) - logX\Big]dN(d\omega) \\ &= (r^{¥} - r^{\$})dt + \sigma^x dW^x - \int_\omega h^x \lambda_t(d\omega)dt - \frac{1}{2}(\sigma^x)^2 dt + \int_\omega \Big[log\big(1 + h^x\big)\Big]dN(d\omega) \\ &= (r^{¥} - r^{\$})dt + \sigma^x dW^x - \frac{1}{2}(\sigma^x)^2 dt + \int_\omega \Big\{log(1 + h^x)dN(d\omega) - h^x\lambda_t(d\omega)dt\Big\} \end{aligned}$$

Intergrate $logX$ from t to T:

$$\begin{aligned} &\int_t^T dlogX^{¥/\$} = logX_T^{¥/\$} - logX_t^{¥/\$} \\ = \quad &\int_t^T (r^{¥} - r^{\$})dt + \int_t^T \sigma^x dW^x - \int_t^T \frac{1}{2}(\sigma^x)^2 dt + \int_t^T\int_\omega \Big\{log\big(1 + h^x\big)dN(d\omega) - h^x\lambda_t(d\omega)dt\Big\} \end{aligned}$$

Take exponential on both sides

$$\begin{aligned} &\frac{e^{-\int_t^T r^{¥}dt} X_T^{¥/\$}}{X_t^{¥/\$}} \\ &= \exp\left\{-\int_t^T r^{\$}dt + \int_t^T \sigma^x dW^x - \int_t^T \frac{1}{2}(\sigma^x)^2 dt + \int_t^T\int_\omega \Big[log(1 + h^x)dN(d\omega) - h^x\lambda_t(d\omega)dt\Big]\right\} \end{aligned}$$

Thus

$$\eta(t,T) = \frac{e^{-\int_t^T r^{¥} dt} X_T^{¥/\$}}{X_t^{¥/\$} e^{-\int_t^T r^{\$} dt}}$$
$$= \exp\left\{\int_t^T \sigma^x dW^x - \int_t^T \frac{1}{2}(\sigma^x)^2 dt + \int_t^T \int_\omega \left[log(1+h^x)dN(dd\omega) - h^x \lambda_t(d\omega)dt\right]\right\} \tag{22}$$

Since Brownian motion and jump process are independent

$$\eta(t,T)$$
$$= \exp\left\{\int_t^T \sigma^x dW^x - \int_t^T \frac{1}{2}(\sigma^x)^2 dt\right\} \exp\left\{\int_t^T \int_\omega \left[log(1+h^x)dN(d\omega) - h^x \lambda_t(d\omega)dt\right]\right\}$$
$$= \eta_{\text{Brownian}} \cdot \eta_{\text{jump}}$$

For Brownian motion, by Girsonov's Therom

$$E^{Q(¥)}[\eta_{\text{diffusion}} | \mathscr{F}_t] = E^{Q(¥)}\left[\exp\left\{\int_t^T \sigma^x dW^x - \int_t^T \frac{1}{2}(\sigma^x)^2 dt\right\} \middle| \mathscr{F}_t\right] = 1$$

For jump process

$$\eta_{\text{jump}} = \exp\left\{log(1+h^x)dN(\omega) - h^x \lambda_t(\omega)dt\right\}$$
$$E^{Q(¥)}\left[f(dN)\eta_{\text{jump}} | \mathscr{F}_t\right]$$
$$= f(0)e^{-h^x \lambda_t(\omega)dt}\left(1 - \lambda_t(\omega)dt\right) + f(1)e^{log(1+h^x) - h^x \lambda_t(\omega)dt} \lambda_t(\omega)dt$$
$$= f(0)\left(1 - h^x \lambda_t(\omega)dt\right)\left(1 - \lambda_t(\omega)dt\right) + f(1)\left(1+h^x\right)\left(1 - h^x \lambda_t(\omega)dt\right)\lambda_t(\omega)dt$$
$$= f(0)\left(1 - (1+h^x)\right)\lambda_t(\omega)dt + f(1)\left(1+h^x\right)\lambda_t(\omega)dt$$
$$= \widetilde{E}[f(dN) | \mathscr{F}_t]$$

Therefore

$$
\begin{aligned}
&E^{Q(¥)}[\eta_{\text{jump}}|\mathscr{F}_t] = 1\\
&E^{Q(¥)}[\eta(t,T)|\mathscr{F}_t] = E^{Q(¥)}[\eta_{\text{Brownian}}|\mathscr{F}_t]\cdot E^{Q(¥)}[\eta_{\text{jump}}|\mathscr{F}_t] = 1
\end{aligned}
$$

Since

$$
e^{adW-\frac{1}{2}a^2dt} = \eta(t,T)_{\text{Brownian motion}} = \exp\left\{\int_t^T \sigma^x dW^x - \int_t^T \frac{1}{2}(\sigma^x)^2 dt\right\}
$$

Thus $a = \int_t^T \sigma^x dt$.

By Girsonov's Theorem for jump process

$$
E[dN\cdot e^{logQdN-(e^Q-1)dt}|\mathscr{F}_t] = \tilde{E}[d\tilde{N}|\mathscr{F}_t]
$$

$$
\eta_{\text{jump}} = \exp\left\{log(1+h^x)dN(d\omega) - h^x\lambda_t^x(d\omega)dt\right\}
$$

Thus $Q = log(1+h^x)$.

Therefore, Dollar interest rate process in Yen risk neutral measure is

$$
\begin{aligned}
df^{\$}(t,T)\Big|^{Q(¥)} &= \vec{\sigma}^{\$}(t,T)\left(\int_t^T \vec{\sigma}^{\$}(\tau,\dot{T})d\dot{T} - \sigma^x\vec{\rho}\right)dt + \vec{\sigma}^{\$}(t,T)d\vec{W}_t^{\$}\\
&+ \int_\omega h^{\$}(t,T)\left\{dN_t(d\omega) - e^{-\int_t^T h^{\$}(t,\dot{T})d\dot{T}+log(1+h^x)}\lambda_t(d\omega)dt\right\}
\end{aligned}
$$

Finally, the standard form of log Radon-Nikodym Derivative P is

$$
\begin{aligned}
& log \frac{e^{-\int_t^T r^{¥} dt} X_T}{X_t D^{\$}(t,T)} \Big|^{Q(¥)} \\
= \; & -\frac{1}{2}\int_t^T \Big[\sigma_\tau^{x2} - 2\sigma_\tau^x \vec{\rho} \int_\tau^T \vec{\sigma}^{\$}(\tau,\dot{T}) d\dot{T} + \Big\| \int_\tau^T \vec{\sigma}^{\$}(\tau,\dot{T}) d\dot{T} \Big\|^2 \\
+ \; & 2\int_\omega \Big((1+h_\tau^x) e^{-\int_\tau^T h^{\$}(\tau,\dot{T})d\dot{T}} - 1 \Big) \lambda_\tau(d\omega) \Big] d\tau \\
+ \; & \int_{\tau=t}^T \sigma_\tau^x dW^x - \int_{\tau=t}^T \int_\tau^T \vec{\sigma}^{\$}(\tau,\dot{T}) d\dot{T} d\vec{W}_\tau^{\$} + \int_\omega \int_{\tau=t}^T \Big(log(1+h_t^x) - \int_t^T h^{\$}(t,\dot{T}) d\dot{T} \Big) dN_\tau(d\omega)
\end{aligned}
$$

4 Exhange Rate future price by a new discount function

Recall

$$
f^{H_T(¥)}(t,T) = E^{H(¥)}\Big[r^{¥} \frac{e^{-\int_t^T r^{¥} dt}}{D^{H_T(¥)}(t,T)} \Big| \mathscr{F}_t \Big] = E^{F_T(\$)}\Big[r^{¥} \Big| \Phi_t \Big]
$$

Set Yen forward rate process also follows HJM, like Dollar forward rate under Dollar risk neutral measure derived in Section 3, Yen forward rate under Yen risk neutral measure is written as

$$
\begin{aligned}
df^{¥}(t,T)\Big|^{Q(¥)} &= \vec{\sigma}^{¥}(t,T) \int_t^T \vec{\sigma}^{¥}(t,\dot{T}) d\dot{T} dt + \vec{\sigma}^{¥}(t,T) d\vec{W}_t^{¥} \\
&+ \int_\omega h^{¥}(t,T) \Big\{ dN_t(d\omega) - e^{-\int_t^T h^{¥}(t,\dot{T})d\dot{T}} \lambda_t(d\omega) dt \Big\} \\
r_T^{¥}|^{Q(¥)} &= f^{¥}(t,T) + \int_t^T \Big(\vec{\sigma}^{¥}(\tau,T) \int_\tau^T \vec{\sigma}^{¥}(\tau,\dot{T}) d\dot{T} - \int_\omega h^{¥}(\tau,T) e^{-\int_\tau^T h^{¥}(\tau,\dot{T})d\dot{T}} \lambda_\tau(d\omega) \Big) d\tau \\
&+ \int_{\tau=t}^T \vec{\sigma}^{¥}(\tau,T) d\vec{W}_\tau^{¥} + \int_{\tau=t}^T \int_\omega h^{¥}(\tau,T) dN_\tau(d\omega)
\end{aligned}
$$

Taking expectation under the Dollar forward measure, and compared to standard form of Radon-Nikodym derivative P derived above

$$
\begin{aligned}
E^{F_T(\$)}\left[r^{¥}\middle|\Phi_t\right] &= f^{H(¥)}(t,T) = f^{¥}(t,T) \\
&+ E^{F_T(\$)}\Big[\int_t^T \vec{\sigma}^{¥}(\tau,T)\Big(\int_{\tau=t}^T \vec{\sigma}^{¥}(\tau,\dot{T})d\dot{T} + \sigma_\tau^x\vec{\rho}_\tau - P_\tau^{¥\otimes\$}\int_\tau^T \vec{\sigma}^{\$}(\tau,\dot{T})d\dot{T}\Big)d\tau|\Phi_t\Big] \\
&+ E^{F_T(\$)}\Big[\int_t^T\int_\omega h^{¥}(\tau,T)\Big((1+h_\tau^x)e^{-\int_\tau^T h^{\$}(\tau,\dot{T})d\dot{T}} - e^{-\int_\tau^T h^{¥}(\tau,\dot{T})d\dot{T}}\Big)\lambda_\tau(d\omega)d\tau|\mathscr{F}_t\Big]
\end{aligned}
$$

Intergrating in terms of T

$$
\begin{aligned}
& D^{H_T(¥)}(t,T) \\
=\ & D^{¥}(t,T)\exp\Big\{-E^{f(\$)}\Big[\int_t^T\Big\{\frac{1}{2}\Big\|\int_\tau^T \vec{\sigma}^{¥}(\tau,\dot{T})d\dot{T}\Big\|^2 \\
+\ & \int_t^T \vec{\sigma}^{¥}(\tau,\ddot{T})\Big(\sigma_\tau^s\vec{\rho}_\tau - P_\tau^{¥\otimes\$}\int_\tau^{\ddot{T}}\vec{\sigma}^{\$}(\tau,\dot{T})d\dot{T}\Big)d\ddot{T} \\
+\ & \int_\omega\Big(e^{-\int_\tau^T h^{¥}(\tau,\dot{T})d\dot{T}} - 1 + \int_t^T h^{¥}(\tau,\ddot{T})(1+h_\tau^x)e^{-\int_\tau^{\ddot{T}} h^{\$}(\tau,\dot{T})d\dot{T}}d\ddot{T}\Big)\lambda_\tau(d\omega)\Big\}d\tau|\mathscr{F}_t\Big]\Big\}
\end{aligned}
$$

Finally, foreign exchange future pricing formula under a new discount function is

$$
\begin{aligned}
& H(t,T) = \frac{X_t D^{\$}(t,T)}{D^{H_T(¥)}(t,T)} \\
=\ & \frac{X_t D^{\$}(t,T)}{D^{¥}(t,T)}\exp\Big\{E^{F_T(\$)}\Big[\int_t^T\Big\{\frac{1}{2}\Big\|\int_\tau^T \vec{\sigma}^{¥}(\tau,\dot{T})d\dot{T}\Big\|^2 \\
+\ & \int_t^T \vec{\sigma}^{¥}(\tau,\ddot{T})\Big(\sigma_\tau^s\vec{\rho}_\tau - P_\tau^{¥\otimes\$}\int_\tau^{\ddot{T}}\vec{\sigma}^{\$}(\tau,\dot{T})d\dot{T}\Big)d\ddot{T} \\
+\ & \int_\omega\Big(e^{-\int_\tau^T h^{¥}(\tau,\dot{T})d\dot{T}} - 1 + \int_t^T h^{¥}(\tau,\ddot{T})(1+h_\tau^x)e^{-\int_\tau^{\ddot{T}} h^{\$}(\tau,\dot{T})d\dot{T}}d\ddot{T}\Big)\lambda_\tau(d\omega)\Big\}d\tau\Big|\mathscr{F}_t\Big]\Big\}
\end{aligned}
$$

5 Conclusion

This paper derives a closed form pricing formula for foreign exchange future, with the spot exchange rate following jump-diffusion process using Interest rate parity, and forward interest rate following HJM model extended to include jumps. In contrast to previous work, this paper aim to allow interest rates to have jumps, correlations between exchange rate and interest rates. With all the complexity in hand, we still can derive pricing formula for future that is reasonably computable.

All the parameters in the pricing formula can be get from market or historical data, except for risk neutral harzard rate λ. We leave this problem to investment banks, insurance companies or anyone who is willing to write out insurance polices for jump events in these three market. In fact, as in financial market, before 2008, there were insurance companies like American International Group (AIG) selling credit risk insurance products.

For the impact of 2008 financial cirsis and tightened regulations on exotic financial products post the crisis, until now there is few insurance product for jump event sold publicly in the financial market. However, for the significance of foreign exchange in financial market, there is definitely a need for the foreign exchange insurance product, thus risk neutral harzard rate can be provided. Indeed, for an financial institution to survive without perfect hedge portfolio in incomplete market, they need to provide the appropriate risk neutral harzard rate λ that covers their operational costs and jump risks. The valuation of this risk neutral harzard rateλ is another complex and interesting problem in reality, which is beyond the scope of this paper.

This paper leaves out empirical research to see whether predited future price in this paper fits the market obeserves better or not compared with other models. This is because the main purpose of this paper is not to derive a more accurate model, but to derive a general method to price asset with jumps and with complex impact from different markets, when arbitrage free condition no longer holds and price can not be expressed by hedge portfolio.

For computational tractability problem, this paper leaves out the pricing formula for foreign exchange option, which may be furture studies for future works.

6 Appendix: Mathematical Tools

6.1 Filtrations

In a continuous-time market, information about the market scenario is revealed at different times. Some events may be determined by the first trading period, others by the second. In this paper $\mathscr{F}_t$ indicates using all historical but not future information available until time t about the stochastic process.

6.2 Cummulant generating Function

For any random variable X~ N(μ, σ), Cumulant generating function

$$
\begin{aligned}
E[e^{\theta X}] &= \int_{-\infty}^{+\infty} e^{\theta x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2 - 2\sigma^2\theta x)\} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2\sigma^2}[x - (\mu + \sigma^2\theta)]^2 + \mu\theta + \frac{1}{2}\sigma^2\theta^2\} dx \\
&= \exp\{\mu\theta + \frac{1}{2}\sigma^2\theta^2\} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}[x - (\mu + \sigma^2\theta)]^2\} dx \\
&= \exp\{\mu\theta + \frac{1}{2}\sigma^2\theta^2\} \cdot \int_{-\infty}^{\infty} f(x) dx
\end{aligned}
$$

Since f(x) is the probability density function for Normal distribution N($\mu + \sigma^2, \sigma^2$)

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x) dx &= 1 \\
E[e^{\theta X}] &= exp\{\mu\theta + \frac{1}{2}\sigma^2\theta^2\}
\end{aligned}
$$

Cumulant Generating Function of X is

$$
\psi_X(\theta) = log(E[e^{\theta x}]) = \mu\theta + \frac{1}{2}\sigma^2\theta^2
$$

6.3 Girsanov Theorem and Radon-Nikodym derivative

Girsanov theorem relates the geometric Brownion motion in probability measure P to different probability measures Q on the space of continuous paths. Assume a geometric Brownian Motion on the probability space P with underlying filtration $\mathscr{F}_t$

$$\begin{aligned}\frac{dS}{S}\bigg|^P &= \mu dt + \sigma_t dW_t \\ &= r dt + \sigma_t (dW_t + \frac{\mu - r}{\sigma_t} dt)\end{aligned}$$

By Girsanov theorem, a geometric Brownian Motion on the new probability space Q is

$$\frac{dS}{S}\bigg|^Q = r dt + \sigma_t d\tilde{W}_t$$

Where

$$E^P[dW_t] = E^P[\eta d\tilde{W}_t] = E^Q[d\tilde{W}_t]$$

η is called the Radon-Nikodym derivative between two measures P and Q and has three properties:

1. $E^P[\eta] = 1$
2. standard form of η is written in function of dW and dt
3. relationship between $d\tilde{W}_t$ and $d\tilde{W}$ can be derived using the standard form of η

6.3.1 Standard form of Radon-Nikodym derivative for Brownian motion

dW_x,dW_y are Brownian motion as $(dW_x, dW_y) \sim N\big(0,\sum dt\big)$, where $\sum$ is the covariance matrix and correlation between dW_1,dW_x is ρ

$$
\begin{aligned}
\Psi_{dW_x,dW_y}(a,b) = logE[e^{adW_x+bdW_y}] &= \frac{1}{2}(a,b)\sum\frac{1}{2}(a,b)^T \\
&= \frac{1}{2}a^2dt+\frac{1}{2}b^2dt+ab\rho dt
\end{aligned}
$$

$$
\begin{aligned}
E[f(dW_y)] &= E[f(0)+f'(0)dW_y+\frac{f''(0)}{2!}dW_y^2+\frac{f'''(0)}{3!}dW_y^3+\cdots] \\
&= E[(1+dW_y\frac{d}{d(dW_y)}+\frac{dW_y^2}{2!}\frac{d^2}{d(dW_y)^2}+\frac{dW_y^3}{3!}\frac{d^3}{d(dW_y)^3}+\cdots)]f(0) \\
&= E[e^{dW_y\frac{d}{d(dW_y)}}f(0)]
\end{aligned}
$$

$$
\begin{aligned}
E[e^{adW_x}f(dW_y)] &= E[e^{adW_x+\frac{d}{d(dW_y)}dW_y}]f(0) \\
&= \exp\Big\{\frac{1}{2}a^2dt+\frac{1}{2}(\frac{d}{d(dW_y)})^2dt+a\frac{d}{d(dW_y)}\rho dt\Big\}f(0) \\
&= e^{\psi_{dW_x}(a)}e^{a\frac{d}{d(dW_y)}\rho dt}E(e^{dW_y\frac{d}{d(dW_y)}})f(0) \\
&= e^{\psi_{dW_x}(a)}e^{a\frac{d}{d(dW_y)}\rho dt}E[f(dW_y)] \\
&= e^{\psi_{dW_x}(a)}E[f(dW_y+aCov(dW_x,dW_y))]
\end{aligned}
$$

1. When $dW_x = dW_y = dW$, and $f(dW) = dW$

$$E[e^{adW}dW] = e^{\psi_{dW}(a)}E[dW+adt]$$

Thus

$$E[(e^{adW-\psi_{dW}(a)})dW] = E[(e^{adW-\frac{1}{2}a^2dt})dW] = E[dW+adt] = \tilde{E}[d\tilde{W}]$$

Standard form of Radon-Nikodym derivative is:

$$e^{adW-\frac{1}{2}a^2dt}$$

Relationship between $d\tilde{W}$ and dW is:

$$d\tilde{W} = dW + adt$$

The above results can be extended to multi-dimentional Brownian motion.

2. When $d\vec{W}_x = d\vec{W}_y = d\vec{W}$ and $f(d\vec{W}) = d\vec{W}$

Standard form of Radon-Nikodym derivative is:

$$e^{\vec{a}d\vec{W}-\frac{1}{2}\vec{a}^2dt} = e^{\vec{a}d\vec{W}-\frac{1}{2}||\vec{a}||^2dt}$$

Relationship between $d\vec{\tilde{W}}$ and $d\vec{W}$ is:

$$d\vec{\tilde{W}} = d\vec{W} + \vec{a}dt$$

3. When $d\vec{W}_y = d\vec{W}$, $dW_x \cdot d\vec{W} = \vec{\rho}dt$ and $f(d\vec{W}) = d\vec{W}$

Standard form of Radon-Nikodym derivative is:

$$e^{adW_x-\frac{1}{2}a^2dt}$$

Relationship between $d\vec{\tilde{W}}$ and $d\vec{W}$ is:

$$d\vec{\tilde{W}} = d\vec{W} + a\vec{\rho}dt$$

6.3.2 Standard form of Radon-Nikodym derivative extended to jump

Standard form of Radon-Nikodym derivative extended to jump can also be derived

$$E[f(dN)e^{\alpha dN+\beta}|\mathscr{F}_t] \quad = \quad \tilde{E}[f(d\tilde{N})|\mathscr{F}_t]$$

For jump process dN under small time period dt

$$dN = \begin{cases} 0, p = 1 - \lambda_t dt = 1 - E[dN|\mathscr{F}_t] \\ \\ 1, p = \lambda_t dt = E[dN|\mathscr{F}_t] \end{cases}$$

Thus

$$\begin{aligned} & f(1)e^{\alpha+\beta}E[dN|\mathscr{F}_t] + f(0)e^{\beta}(1 - E[dN|\mathscr{F}_t]) \\ = \quad & f(1)\tilde{E}[d\tilde{N}|\mathscr{F}_t] + f(0)(1 - \tilde{E}[d\tilde{N}|\mathscr{F}_t]) \end{aligned}$$

Compare parameters before f function

$$\begin{aligned} e^{\alpha+\beta}E[dN|\mathscr{F}_t] \quad &= \quad \tilde{E}[d\tilde{N}|\mathscr{F}_t] \\ e^{\beta}(1 - E[dN|\mathscr{F}_t]) \quad &= \quad 1 - \tilde{E}[d\tilde{N}|\mathscr{F}_t] \end{aligned}$$

Since $E[f(dN)|\mathscr{F}_t] = \lambda dt \propto dt$, $\tilde{E}[f(d\tilde{N})|\Phi_t] = \tilde{\lambda} dt \propto dt$

$$\begin{aligned}
\beta &= log\frac{1-\tilde{E}[d\tilde{N}|\mathscr{F}_t]}{1-E[dN|\mathscr{F}_t]} = \tilde{E}[d\tilde{N}|\mathscr{F}_t] - E[dN|\mathscr{F}_t] = (1-Q)\lambda_t dt \\
\alpha &= log\frac{\tilde{E}[d\tilde{N}|\mathscr{F}_t]}{E[dN|\mathscr{F}_t]} = logQ
\end{aligned}$$

Thus standard form of Radon-Nikodym derivative for jump process is

$$E[f(dN)e^{logQdN+(1-Q)\lambda_t dt}|\mathscr{F}_t] \quad = \quad \tilde{E}[f(d\tilde{N})|\mathscr{F}_t]$$

6.3.3 Independent nature of Radon-Nikodym derivative for Brownian motion and jump

By Levy Ito's Decomposition, Brownian motion and jump process are independent. For Brownion motion, change of measure means defining a new Brownion motion with different dirft term. But for jump process, dN is 0 or 1 regardless of the measure. Here we want to prove that Brownian motion and jump process can be treated seperately when chaning measure, and Radon-Nikodym derivative $\eta_{\text{jump-diffusion}}$ for a jump-diffusion process is a multiplication of Radon-Nikodym derivative η_{Brownian} for Brownian motion, and Radon-Nikodym derivative η_{Jump} for jump process.

To prove Brownian motion and jump process can be treated seperately:

set $X_1 = dW$and $X_2 = dN$

$$\begin{aligned}
& E\Big[f(uX_1 + vX_2)e^{\alpha X_1 + \beta X_2 + \gamma dt}\Big|\mathscr{F}_t\Big] \\
= \quad & E\Big[E\big[f(uX_1 + vX_2)e^{\alpha X_1 + \gamma_0 dt}\big|X_2\big]e^{\beta X_2 + (\gamma - \gamma_0)dt}\Big|\mathscr{F}_t\Big]
\end{aligned}$$

Thus proved Brownian motion and jump process can be treated seperately by taking conditional expectations. Now set a jump-diffusion process as

$$\vec{X} = \int \vec{\sigma} d\vec{W} + \int_{\omega} \vec{h} dN(d\omega) + \int \vec{\theta} dt$$

Thus

$$\begin{aligned}
& \tilde{E}[\tilde{f}(X)|\Phi_t] = E[f(X)\eta|\mathscr{F}_t] \\
= \quad & E\Big[f\big(dW_1...dW_n, dN(d\omega_1)...dN(d\omega_n)\big)e^{\lambda_1 dt+\alpha_1 dW_1}e^{\lambda_2 dt+\alpha_2 dW_2}... \\
& e^{u_1 dt+\beta_1 dN(d\omega_1)}e^{u_2 dt+\beta_2 dN(d\omega_2)}...\Big|\mathscr{F}_t\Big] \\
= \quad & E\Big[f\big(dW_1...dW_n, dN(d\omega_1)\eta_{\text{Brownian}} \cdot \eta_{\text{Jump}}|\mathscr{F}_t\Big] \\
= \quad & E\Big[f\big(dW_1...dW_n, dN(d\omega_1)\eta_{\text{jump-diffusion}}|\mathscr{F}_t\Big]
\end{aligned}$$

7 Reference

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